Coalitional Game Theory

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Why do we need Game Theory?

- most of the business enterprises on the internet can be analyzed using game theory
- game-theoretical way of thinking is fundamental in auctions
- do not underestimate game theory!
- do not overestimate it!
Classification of Games

Classification w.r.t. possibility to cooperate:
- Cooperative games - players are allowed to form coalitions, share payoffs
- Non-cooperative games - players are not allowed to form coalitions;

Classification w.r.t. timing of moves:
- Simultaneous-move games;
- Sequential-move games;

Classification w.r.t. information:
- Games with perfect/imperfect information;
- Games with complete/incomplete information;
Part III: Coalitional Games
Assumptions in Non-Cooperative Games

- Cooperation can’t occur in the prisoner’s dilemma because the conditions required for cooperation are not present, in particular:
  - utility is given directly to individuals as a result of individual action;
  - binding agreements are not possible.

- But suppose we drop these assumptions?
Coalitional Games

Coalitional games model scenarios where:

- agents can benefit by cooperating;
- binding agreements are possible;
- benefit/utility is attributed to a collective, not to an individual.
Issues in coalitional games (Sandholm et al, 1999):

- Coalition structure generation.
- Teamwork.
- Dividing the benefits of cooperation.
Coalition Structure Generation

- Deciding in principle who will work together.
- The basic question:
  
  Which coalition should I join?

- The result: partitions agents into disjoint coalitions. The overall partition is a coalition structure.
Solving the optimization problem of each coalition

- Deciding how to work together.
- Solving the “joint problem” of a coalition
- Finding how to maximise the utility of the coalition itself.
- Typically involves joint planning etc.
Dividing the Benefits

- Deciding “who gets what” in the payoff.
- Coalition members cannot ignore each other’s preferences, because members can defect: if you try to give me a bad payoff, I can always walk away.
- We might want to consider issues such as fairness of the distribution.
Coalitional Games

A coalitional game is a pair:

\[ \langle Ag, \nu \rangle \]

where:

- \( Ag = \{1, \ldots, n\} \) is a set of players or agents;
- \( \nu : 2^{Ag} \rightarrow \mathbb{R} \) is the characteristic function of the game.

Usual interpretation: if \( \nu(C) = k \), then coalition \( C \) can cooperate in such a way they will obtain utility \( k \), which may then be distributed amongst team members.
Most important question in coalitional games: is a coalition stable?

that is,

is it rational for all members of coalition to stay with the coalition, or could they benefit by defecting from it?

(There is no point in me trying to join a coalition with you unless you want to form one with me, and vice versa.)

Stability is a necessary but not sufficient condition for coalitions to form.
The core of a coalitional game is the set of feasible distributions of payoff to members of a coalition that no sub-coalition can reasonably object to.

An outcome for a coalition $C$ in game $\langle Ag, \nu \rangle$ is a vector of payoffs to members of $C$, $\langle x_1, \ldots, x_k \rangle$ which represents a feasible distribution of payoff to members of $Ag$.

“Feasible” means:

$$\nu(C) \geq \sum_{i \in C} x_i$$

Example: if $\nu(\{1, 2\}) = 20$, then possible outcomes are $\langle 20, 0 \rangle, \langle 19, 1 \rangle, \langle 18, 2 \rangle, \ldots, \langle 0, 20 \rangle$. (Actually there will be infinitely many!)
Objections

- Intuitively, a coalition $C$ objects to an outcome if there is some outcome for them that makes all of them strictly better off.

- Formally, $C \subseteq Ag$ objects to an outcome $\langle x_1, \ldots, x_n \rangle$ for the grand coalition if there is some outcome $\langle x'_1, \ldots, x'_k \rangle$ for $C$ such that
  
  $$x'_i > x_i \quad \text{for all } i \in C$$

- The idea is that an outcome is not going to happen if somebody objects to it!
The Core

- The **core** is the set of outcomes for the **grand coalition** to which no coalition objects.
- If the core is **non-empty** then the **grand coalition is stable**, since nobody can benefit from defection.
- Thus, asking

  is the grand coalition stable?

is the same as asking:

is the core non-empty?
Sometimes, the core is empty; what happens then?

Sometimes it is non-empty but isn’t “fair”.

Suppose \( \text{Ag} = \{1, 2\} \), \( \nu(\{1\}) = 5 \), \( \nu(\{2\}) = 5 \), \( \nu(\{1, 2\}) = 20 \).

Then outcome \( \langle 20, 0 \rangle \) (i.e., agent 1 gets everything) is not in the core, since the coalition \{2\} can object. (He can work on his own and do better.)

However, outcome \( \langle 15, 5 \rangle \) is in the core: even though this seems unfair to agent 2, this agent has no objection.

Why unfair? Because the agents are identical!
The Shapley value $\varphi_i$ for agent $i$ is best known attempt to define how to divide benefits of cooperation fairly.

It does this by taking into account how much an agent contributes.

The Shapley value of agent $i$ is the average amount that $i$ is expected to contribute to a coalition.

Axiomatically: a value which satisfies axioms: symmetry, dummy player, and additivity.
Shapley’s Axioms: Symmetry

- The symmetry axiom says that agents which make the same contribution should get the same payoff.
- Let $\delta_i(S)$ be the amount that $i$ adds by joining $S \subseteq Ag$:

$$\delta_i(S) = \nu(S \cup \{i\}) - \nu(S)$$

... the marginal contribution of $i$ to $S$.

- Then $i$ and $j$ are interchangeable if $\delta_i(S) = \delta_j(S)$ for every $S \subseteq Ag \setminus \{i, j\}$.
- The symmetry axiom: if $i$ and $j$ are interchangeable then $\phi_i = \phi_j$. 

The dummy player axiom says that agents which make no contribution should get what they could get on their own.

Formally, $i \in Ag$ is a dummy if $\delta_i(S) = v(\{i\})$ for every $S \subseteq Ag \setminus \{i\}$.

The dummy player axiom: if $i$ is a dummy player then $\varphi_i = v(\{i\})$. 
Shapley’s Axioms: Additivity

- This one is a bit technical! It basically says that if you combine two games, then the value a player gets should be the sum of the values in the individual games.
- You can’t gain or lose by playing more than once.
The Shapley value for $i$, denoted $\varphi_i$, is:

$$
\varphi_i = \sum_{r \in R} \delta_i(S_i(r)) \frac{1}{|Ag|!}
$$

where $R$ is the set of all orderings of $Ag$ and $S_i(r)$ is the set of agents preceding $i$ in ordering $r$.

This is the unique solution to Shapley’s axioms.
Part IV - Computational Issues
It is important for an agent to know (eg) whether the core of a coalition is non-empty . . . so, how hard is it to decide this?

Problem: naive, obvious representation of coalitional game is \textit{exponential} in the size of $Ag$!

Now such a representation is:

- \textit{utterly} infeasible in practice; and
- so large that it renders comparisons to this input size meaningless: stating that we have an algorithm that runs in (say) time \textit{linear} in the size of such a representation means it runs in time \textit{exponential} in the size of $Ag$!
How to Represent Characteristic Functions?

Two approaches to this problem:

- try to find a complete representation that is succinct in “most” cases
- try to find a representation that is not complete but is always succinct
- A common approach: interpret characteristic function over combinatorial structure.
Part V - Other Representations
The Induced Subgraph Representation

- Represent $\nu$ as an undirected graph on $Ag$, with integer weights $w_{i,j}$ between nodes $i, j \in Ag$.
- Value of coalition $C$ then:

$$\nu(C) = \sum_{\{i,j\} \subseteq Ag} w_{i,j}$$

i.e., the value of a coalition $C \subseteq Ag$ is the weight of the subgraph induced by $C$.

![Diagram illustrating the original graph and the induced subgraph for coalition (A,B,C).]
Complexity of Induced Subgraphs

(Deng & Papadimitriou, 94)

- Computing Shapley: in polynomial time. An agent get’s half the income from its edges.

\[ \varphi_i = \frac{1}{2} \sum_{j \neq i} w_{i,j} \]

- However, determining emptiness of the core is \text{NP-complete}

- Checking whether a specific distribution is in the core is \text{co-NP-complete}

This representation is not complete.
Marginal Contribution Nets

(leong & Shoham, 2005)

- Characteristic function represented as rules:

  \[
  \text{pattern} \rightarrow \text{value}.
  \]

- Pattern is conjunction of agents, a rule applies to a group of agents \( C \) if \( C \) is a superset of the agents in the pattern.

  Value of a coalition is then sum over the values of all the rules that apply to the coalition.

Example:

\[
\begin{align*}
a \land b & \rightarrow 5 \\
b & \rightarrow 2
\end{align*}
\]

We have: \( \nu(\{a\}) = 0, \nu(\{b\}) = 2, \) and \( \nu(\{a, b\}) = 7. \)

- We can also allow negations in rules (agent not present).
Marginal Contribution Nets

- Shapley value: in polynomial time. Consider case where rules only contain positive literals, let $\rho$ be set of such rules representing a game. Then:

$$\varphi_i = \sum_{r \in \rho: i \text{ in lhs of } r} \varphi_i^r$$

where

$$\varphi_i^{\chi \rightarrow x} = \frac{x}{|\chi|}$$

- Checking whether distribution is in the core is co-NP-complete.
- Checking whether the core is non-empty is co-NP-hard.

A complete representation, but not necessarily succinct.
Dominant strategies

- In some situations a player might have one best strategy, irrelevant what other players choose;
- Then we say that such a strategy is strongly (or strictly) dominant for a player;
- Rational player will always choose his dominant strategy if he has one;
- If every player had a dominant strategy, predicting the outcome of the game is a trivial exercise;
- However, it is rarely a case.
Definition. Consider two pure strategies for player $i$: $s_i, s'_i \in S_i$. We say that $s'_i$ strictly dominates $s_i$ if $s'_i$ gives player $i$ a strictly higher expected utility than does $s_i$ for every possible deleted pure-strategy profile $s_{-i} \in S_{-i}$ which her opponents could play. More formally:

$$\forall s_{-i} \in S_{-i} \quad u_i(s'_i, s_{-i}) > u_i(s_i, s_{-i})$$
Pure-strategy strong dominance - Example

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Definition. If each player $i \in N$ has a dominant strategy $s^*_i$ than the game solution is the strategy combination:

$$s^* = \{s^*_1, \ldots, s^*_n\}$$

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<td>$s^2_2$</td>
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Game of Matching Pennies

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- **a**: 1, -1, -1, 1
- **b**: -1, 1, 1, -1

Does any player have a dominant strategy? The answer is "no".

Conclusion: Not all games are dominance solvable.
Game of Matching Pennies

- Does any player have dominant strategy?
Game of Matching Pennies

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- Does any player have dominant strategy?
- The answer is "no"
Game of Matching Pennies

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- The answer is "no"
- Conclusion: Not all games are dominance solvable.
What do rationality and common knowledge assumptions mean?

What does it mean that the strategies, payoffs, etc. are of common knowledge? Let $P$ be a certain fact which is of common knowledge.
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- Everyone knows \( P \);
What do rationality and common knowledge assumptions mean?

What does it mean that the strategies, payoffs, etc. are of common knowledge? Let $P$ be a certain fact which is of common knowledge.

- Everyone knows $P$;
- Everyone knows that (Everyone knows $P$);
What do rationality and common knowledge assumptions mean?

What does it mean that the strategies, payoffs, etc. are of common knowledge? Let $P$ be a certain fact which is of common knowledge.

- Everyone knows $P$;
- Everyone knows that (Everyone knows $P$);
- Everyone knows that [Everyone knows that (Everyone knows $P$)];
- Etc.
Iterated dominance - Example

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The game shows the payoffs for each player in each strategy combination. The iterated dominance process is indicated by the red lines and the blue circle, which highlight the dominated strategies and the resulting Nash equilibrium.
Iterated dominance - Experiment Example

Ft-data (1468 subjects)

average 18.91, winning number: 13
Solve this one

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Does the order matter?
Solve this one

Does the order matter?
For *strict dominance* not
Solve this one

Does the order matter?
For **strict dominance** not
For **weak dominance** yes (in general)

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